Problem Sheet 4

In this problem sheet, unless otherwise stated, for a Gaussian measure μ on $\mathbb{R}^n$ we fix $m \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ such that for all $\lambda \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} \mu(dx) = e^{i\langle \lambda, m\rangle - \frac{1}{2}\langle K\lambda, \lambda\rangle}. \tag{1}$$

We also define the Fourier Transform of a Borel Measure ν on $\mathbb{R}^n$ by

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \nu(dx) \tag{2}$$

and for ν on a Banach Space E, $\hat{\nu} : E^* \to \mathbb{C}$, by

$$\hat{\nu}(l) = \int_E e^{il(x)} \nu(dx).$$

Exercise 4.1.

For $y \in \mathbb{R}^n$, prove that δ_y is a Gaussian measure. If $y \in E$ for a Banach Space E, is δ_y a Gaussian measure?

Exercise 4.2.

Let μ, ν be probability measures on a separable Banach Space E.

1. Show that if $l_*\mu = l_*\nu$ for all $l \in E^*$, then $\mu = \nu$.

   Hint: As a consequence of the Hahn-Banach Separation Theorem, every closed ball $B \subset E$ admits the representation $B = \bigcap_{i\in I} A_i$ for some countable indexing set I, and for sets A_i of the form $A_i = \{x \in E : l(x) \leq c\}$.

2. Prove that if $\hat{\mu}(l) = \hat{\nu}(l)$ for all $l \in E^*$, then $\mu = \nu$.

Exercise 4.3.

1. Let $\{X_1, \ldots, X_N\}$ be independent random variables such that each X_j is Gaussian on $\mathbb{R}^n$, and $a_j \in \mathbb{R}$. Show that $\sum_{j=1}^N a_j X_j$ is Gaussian on $\mathbb{R}^n$.
2. Let $\{X_1, \ldots, X_N\}$ be independent random variables such that each X_j is Gaussian on $\mathbb{R}$, and $a_j \in E$ for some Banach Space E. Show that $\sum_{j=1}^N a_j X_j$ is Gaussian on E.

Exercise 4.4.

Consider a sequence of real numbers (ε_n) convergent to zero as $n \to \infty$, and corresponding Gaussian measures μ_n on $\mathbb{R}$ with mean m and variance ε_n^2. Prove that (μ_n) converges weakly to δ_m as $n \to \infty$.

Exercise 4.5.

Let (e_j) be a basis of a separable Hilbert Space H, (Y_j) a collection of independent standard real-valued Gaussian random variables (mean zero and variance one), and $X_n = \sum_{j=1}^n e_j Y_j$. We use $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ for the inner product and norm on H, respectively.

1. Show that the sequence of real-valued functions $(\widehat{\mu_n}(\cdot))$ on H converges pointwise to $e^{-\frac{1}{2}\|\cdot\|^2}$.
2. Prove that Lévy's Continuity Theorem, Theorem 2.4.6, does not hold if one replaces $\mathbb{R}^d$ by a general separable Hilbert Space H.

Exercise 4.6

Let T be a positive symmetric linear operator on a separable Hilbert Space. Prove that its trace as in Definition 2.6.6 is independent of the choice of basis.